A Framework for Mathematicians' Example-Related Activity When Exploring and Proving Mathematical Conjectures

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Examples play a critical role in mathematical practice, particularly in the exploration of conjectures and in the subsequent development of proofs. Although proof has been an object of extensive study, the role that examples play in the process of exploring and proving conjectures has not received the same attention. In this paper, we present a framework that characterizes ways in which mathematicians utilize examples when investigating conjectures and developing proofs. The data consist of 133 mathematicians' responses to two open-ended survey questions. The framework offers categories for the types of examples, uses of examples, and example strategies that mathematicians discussed in reference to their work with conjectures. In addition to presenting the framework, we also discuss potential educational implications of the results.

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Introduction

A perennial concern in mathematics education is that students fail to understand the nature of evidence and justification in mathematics (Kloosterman & Lester, 2004). Mathematics education scholars have suggested that students' struggles with understanding the nature of evidence and justification may be due, in large part, to their views concerning the role of examples; in particular, students tend to be overly reliant on examples and often infer that a (universal) mathematical statement is true on the basis of checking a number of examples that satisfy the statement (e.g., Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Porteous, 1990). One approach designed to help students overcome their overreliance on examples is to help them understand the limitations of examples as a means of justification and thus appreciate the need for a proof (e.g., Harel & Sowder, 1998; Stylianides & Stylianides, 2009; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, in press). Although such an approach may indeed help students understand the limitations of example-based reasoning as well as appreciate the need for proof, it characterizes example-based reasoning strategies as obstacles to overcome. Given the essential role examples play in the exploration of conjectures and in subsequent proof attempts, we suggest that example-based reasoning strategies should not be positioned only as barriers. The field may benefit from a greater understanding of the ways in which those who are adept at proof, such as mathematicians, leverage examples in order to support their thinking and activity.

Although the role of examples in learning mathematics has received attention in the literature (cf., Bills & Watson, 2008), considerably less attention has been directed toward the specific roles examples play in exploring and proving conjectures. In this paper, we present a framework that serves to characterize the variety of ways in which examples arise in mathematicians' exploration of conjectures and development of proofs. In particular, the framework characterizes the *types* of examples mathematicians may choose, the ways in which they may *use* examples, and their described *strategies* for utilizing the examples. We also discuss potential implications of this work for the teaching and learning of proof in school mathematics.

The Interplay between Example-based Reasoning and Proof

Epstein and Levy (1995) contend that "Most mathematicians spend a lot of time thinking about and analyzing particular examples," and they go on to note that "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Clearly, examples play a critical role not only in mathematicians' development of and exploration of conjectures, but also in their subsequent development of proofs of those conjectures. Indeed, there is often a back-and-forth interplay between mathematicians' example-based reasoning activities and their deductive reasoning activities (e.g., Alcock & Inglis, 2008). Several mathematics education researchers have accordingly examined various aspects of the interplay between example-based reasoning activities and deductive reasoning activities among both mathematicians and mathematics students (e.g., Buchbinder & Zaslavsky, 2009; Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Knuth, Choppin, & Bieda, 2009).

Antonini (2006), for example, asked advanced mathematics graduate students to generate examples with specific mathematical properties. The results of this work led to an initial categorization of the students' strategies for producing examples. Antonini hypothesized that the categorized strategies may play an important role in the production of conjectures and proofs. Building upon this research, Iannone et al. (2011) used Antonini's framework to categorize the strategies undergraduate mathematics students used to generate examples during their attempts to produce proofs of conjectures. They were surprised to find that example generation did not seem to have a positive effect on proof production tasks, and they called for more research to be done in studying examples. Finally, Buchbinder and Zaslvsky (2009) also provided a framework for categorizing high school mathematics students' uses of examples when evaluating the validity of mathematical statements; they specifically classified examples as confirming, non-confirming, contradicting, and irrelevant. Although the preceding studies did not focus explicitly on the role examples play in exploring conjectures and developing proofs (or counterexamples), the research underscores the nature of the interplay between example-based reasoning activities and deductive reasoning activities, and it thus serves to inform the research presented in this paper.

Methods

Participants consisted of 133 mathematicians who responded to an on-line survey sent to the mathematics departments at 27 U. S. universities. The focus of this paper is on these experts' responses to the following open-ended prompt: *If you sometimes use examples when exploring a new mathematical conjecture, how do you choose the specific examples you select in order to test or explore the conjecture? What explicit strategies or example characteristics, if any, do you use or consider?* Approximately 58% of the experts were completing PhD's in mathematics, 30% had PhD's in mathematics (in a variety of different mathematical areas), and 12% had advanced degrees in other STEM-related fields; 67% were male. The data consist of these mathematicians' self-reported responses about their work with examples; while we acknowledge limitations to such data, they led us to an initial framework for experts' example-related activity.

Members of the research team independently examined the expert responses to the questions with the intent of identifying the various types of examples the mathematicians reported. During this initial coding of the data, however, it became clear that *types* of examples alone did not sufficiently capture the richness of the responses; in particular, we also coded the data with respect to the mathematicians' *uses* of examples and their *strategies* for using examples. After codes for types, uses, and strategies emerged, two members of the research team re-coded all of the responses; any discrepancies were resolved through discussion with the entire research team.

It is important to note that a particular response often could be coded in multiple ways simultaneously, both within a category (e.g., receiving multiple example-type codes), and across categories (e.g., coded as a particular example-type and as a particular use of examples). For instance, the response "I first do examples that are easiest to test. If those are consistent with the conjecture, I try more general examples, focusing on those for which the conjecture might fail." received the following codes (defined in the following section): ***Types****: Easy to Compute, General/Generic, Counterexample/Conjecture Breaking;* ***Uses****: Check, Break the Conjecture;* ***Strategies****: Multi-Stage Example Exploration: Increasing in Generality.* The total frequencies thus do not necessarily sum to 133, the total number of respondents to the prompt.

Results

In the three sub-sections that follow, we present the components of the framework that characterize the mathematicians' example-related activities when exploring and proving conjectures. Given the page limitations of the conference proceedings, we do not go into great detail about the framework; however, we do provide representative verbatim data excerpts to illustrate the various framework categories (italics in the excerpts indicate researchers' rationale for the respective codes). We examine the results further in the Discussion section.

Types of Examples

Mathematicians described a variety of types of examples that they use when exploring conjectures (Table 1). *Simplicity* was the most frequent type of example discussed by the experts, and they also often considered counterexamples and complex examples in their work.

Table 1: Types of Examples.

Type (Frequency)	Definition	Representative Data Excerpt
Simplicity (72)	Expert appeals to an easy, simple or basic example. Includes "trivial" and "small."	*Easy* ones! Start with toy cases and slowly build up the complexity.
Counterexample /Conjecture Breaking (36)	Expert picks an example that might disprove the conjecture. The expert might explicitly say "a counterexample," but this can also be inferred.	Likewise, I might also check *an example for which I believe the conjecture is most likely to fail* (sort of like a stress test).
Complex (36)	Expert picks a complex example in order to test whether the conjecture holds for tricky ones; synonyms include "non-nice," "non-trivial," or "interesting."	I try to find examples that include all of the (foreseen) barriers to a proof. *The "hardest" examples in the sense of what I'm trying to prove.*
Easy to Compute (32)	Expert chooses an example that is easy to manipulate. The difference between this code and "Simple" is that the expert says something about computing or working the example out.	I usually use appropriate low-level examples. For example, those that may be *easy to compute* and/or for which it is reasonable to check the conjecture.
Properties (26)	Expert takes into account some specific mathematical property – he or she might reference a "property" or "features," or might mention particular properties.	… For number-based conjectures, *I choose 0, numbers close to 0 (both positive and negative), very large and very small numbers, for examples, both integers and non-integers.*
General/Generic	Expert states that he or she uses general or	Try the *most general example* which is still

(22)	generic examples, or describes examples that are viewed as representative of a general class of cases or otherwise lack special properties.	practical to test.
Boundary Case (19)	Expert picks an extreme example or number, or a "special" case, such as the identity.	…I will next try to test some strange or pathological examples, *to really push the boundaries of what might be possible in this situation*.
Familiar/Known case (18)	Expert chooses an example with which he or she is familiar, or in which properties related to the conjecture are already known.	Use examples *I'm familiar with* and see if everything still holds.
Unusual Examples (13)	Expert picks an unusual number, which would be described as something that does not come up often. "Rare," "obscure," "strange," and "weird" are also synonyms.	Next, I try something slightly more *obscure*.
Random (10)	Expert describes the example as randomly chosen; this includes genuine mathematical randomness, such as cases in which examples are chosen with a random number generator.	Try *"random"* examples in cases where that makes sense.
Exhaustive (9)	Expert looks for "all" of the examples in an exhaustive manner. This can be by testing all possible examples or by using a computer.	If it is difficult to find examples, write a computer program to find *all examples with specific characteristics*.
Common (9)	Expert describes the example as typical, common, or one many would choose.	Ones that are not special, *ones that I judge to be typical.*
Dissimilar Set (9)	Expert indicates that he or she purposely selects a variety of different types of examples.	If there are too many, I would try to select examples with *widely different properties*.

Uses of Examples

Table 2 highlights the ways in which the mathematicians discussed how they use examples as they examine conjectures; these ranged from using an example to check whether a conjecture is true to carefully selecting examples that might provide insight into how to prove the conjecture.

Table 2: Uses of Examples.

Use (Frequency)	Definition	Representative Data Excerpt
Check (41)	Expert selects examples to make a judgment about the correctness of a conjecture; "test," "verify," and "check" are all synonyms.	I would start with an example that is easy to *check*. If that example works out and agrees with the conjecture I would *check* a less trivial example then.
Break the Conjecture (35)	Expert tries examples to break the conjecture; this can include specifically looking for a counterexample.	Then once I'm more comfortable with it, *try it with some example I regard as less likely to verify the conjecture and keep looking for a counter-example.*
Make Sense of the Situation (16)	Expert uses an example to deepen his or her understanding of why the conjecture might be true or false, or to gain mathematical insight.	First test for the most simple cases, *also to understand the conjecture a little better*. …
Proof Insight (8)	Expert indicates that his or her production of examples (or counterexamples) might have a direct bearing on understanding how to prove the conjecture.	Eventually, if I figure out the conjecture is true for all the examples tested, the search for a counter-example *should have given me some insight in how to prove it.*
Generalize (5)	Expert mentions using the example to generalize or to allow the expert to work in a more general situation.	… Hopefully it's obvious why what you're looking for is true in the easiest case. *You can then see if that reason generalizes.*

Understand Statement of the Conjecture (3)	Expert uses an example to better understand the statement of the conjecture.	I use simple examples first, *so I understand what the conjecture says* and then build up to more complicated ones.

Example Strategies

Table 3 displays the strategies that mathematicians said they employ when using examples to explore and/or prove conjectures. In order to be coded as a strategy, a mathematician's response had to explicitly describe a systematic approach for how he or she used examples when exploring a conjecture; simply listing one or two actions one would take or examples one would try did not constitute a strategy unless the actions or examples were explicitly connected. The *Multi-Stage Example Exploration* strategy occurred when mathematicians indicated a progression in their choices of examples; typically they described starting with simple examples, and moved toward more complex, more general, or more extreme examples. The *Property Analysis* strategy involved an examination of particular properties of the chosen examples, and insights about these properties provided further insight into the conjecture or the proof. In the *Analysis of Related Proof Activities* strategy, mathematicians described engaging in proof activities that subsequently affected how they chose and used examples. In these cases, the mathematicians' choices and uses of examples were explicitly linked to attempts to prove or to disprove the conjecture. Finally, the *Systematic Variation* strategy occurred when mathematicians suggested that they would start with an example, but would then carefully modify it in some way to further their progress in exploring or proving a conjecture.

Table 3: Example Strategies.

Strategy (Frequency)	Subcategory (Frequency)	Definition	Representative Example
Multi-Stage Example Exploration (62)	*Increasing in Complexity (29)*	Expert begins with simple or easy examples and shifts to more complex or complicated examples.	*I use simple examples first*, so I understand what the conjecture says *and then build up to more complicated ones.*
	Increasing in Extremity (17)	Expert begins with simple or typical examples and builds to boundary cases, special cases, or conjecture-breaking cases.	*First look for the easiest examples. Build sophistication, and look for extremal cases.*
	Increasing in Generality (16)	Expert begins with simple or special examples and shifts to more general or generic examples.	Following Polya's suggestions, *looking at simplest examples first. Then try to choose a few representative/generic/random examples.*
Property Analysis (12)	*Supporting or Non-Supporting Examples (11)*	Expert attempts to determine the properties of examples that either support the conjecture in particular ways, or to determine the properties of examples that do not support the conjecture.	I try *to find specific properties* of my guess examples that prevent them from doing what I want them to do. *Sometimes this allows a slow building up of properties that can eventually say something useful about the conjecture.*
	Test Cases (1)	Expert analyzes special test cases depending on the critical properties of the examples related to the conjecture.	Also, depending on what I think will be important in proving (or disproving) the conjecture, *I will include some special test cases.*

Analysis of Related Proof Activities (11)	*Generalize From Attempts to Prove (6)*	Expert proves related conjectures or lemmas and attempts to generalize to the conjecture at hand. Or, proves the conjecture for specific examples and then attempts to build to a general proof. Or, attempts to generalize from examples related to unsuccessful proof attempts.	Based…*on an unsuccessful proof of the conjecture* (the point where I got stuck may give me suggestions of where to look for negative examples).
	Generalize From Attempts to Disprove (5)	Expert attempts to generalize from examples that disprove related conjectures. Or, attempts to generalize from properties of examples that failed to disprove the conjecture.	I also *try to guess counterexamples*. This guessing typically fails, and if it does, *I try to find specific properties of my guess examples that prevent them from doing what I want them to do.*
Systematic Variation (10)	*Known Case Adjustment (7)*	Expert takes a known case and makes small adjustments to the example's properties, inputs, or characteristics.	Take an example *I've already done and perturb it a bit.*
	Multiple Property Variation (3)	Expert varies multiple properties or characteristics simultaneously or independently in a systematic fashion.	If there are *multiple properties, I often try to vary them independently* to see if I can discover their individual effects.

Discussion

The three-part framework highlights the complex roles examples play in the work of mathematicians, identifying multiple types of examples, uses of examples, and example-based reasoning strategies that mathematicians take into account as they engage in exploring and proving conjectures. What may not necessarily be evident in the presentation of the framework, however, is the ways in which the mathematicians' extensive domain knowledge plays a critical role in conjecture-related activity. Through our analysis, we identified four ways in which mathematicians' domain expertise appears to influence both their example choices and the ways in which they think strategically with examples when making sense of conjectures.

First, mathematicians noted that their approaches to exploring and proving a conjecture were dependent on whether or not they thought the conjecture was true. Their initial instincts about a conjecture's likelihood to be valid influenced the types of examples they chose and the strategies they employed. For example, one expert said, "If I am not sure whether the conjecture is true, I start by considering an example for which I believe it will be true…If I think it is not true, I choose the simplest example for which I believe the conjecture will fail. If I am fairly certain it is true, I usually try to consider the most general case." Second, mathematicians indicated that for many conjectures, constructing an example was not necessarily trivial (or even possible) in certain mathematical domains. One expert said, "In my work…the invariants I work with are incredibly hard to compute," and another noted that, "In my mathematical experience, the trick is to FIND examples" (expert's emphasis). Third, mathematicians implied that prior experience and intuition often played a part in their choice of examples. They described ways in which they capitalize on their prior experiences with a given area in order to access examples that would be most relevant to particular types of conjectures. One expert described choosing examples, "Based on the intuition, on the experience (examples already present in the literature may give a feeling)."

Finally, the mathematicians exhibited a meta-awareness in which they were able to see their example-related activity in terms of a broader context of their mathematical activity. In other words, the mathematicians showed intentionality about their work with examples—they were aware of what their examples could do for them and were often explicitly deliberate about their example choices. In the following response, the expert displays a clear strategy and cognizance when it comes to choosing examples when exploring a conjecture: "First test for the most simple cases, also to understand the conjecture a little better. Then once I'm more comfortable with it, try it with some example I regard as less likely to verify the conjecture and keep looking for a counter-example. Eventually, if I figure out the conjecture is true for all the examples tested, the search for a counter-example should have given me some insight in how to prove it."

The role that examples play in the work of mathematicians stands in contrast to the role examples typically play in the work of mathematics students. For instance, some studies suggest that experts' meta-awareness of examples described above differs from students' example use. Knuth et al. (2011) suggest that middle school students may have difficulty considering the characteristics of their examples in the way mathematicians do. Additionally, while other studies have shown that, like mathematicians, students make use of similar types of examples (such as *Simple, Common*, or *Unusual*) (Cooper, et al., 2011), the ways in which students and mathematicians appear to use examples may differ. Using examples to check a conjecture's accuracy and then as a justification of its truth is common in student populations (Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Porteous, 1990), whereas mathematicians indicated that they rely on examples not only to check conjectures, but also to better understand them and to gain insights into their proofs.

The domain expertise that the mathematicians possess also clearly contributes to the differences between the roles examples play in the work of mathematicians and in the work of students. In the latter case, for example, the majority of conjectures with which students are charged with proving are true (and students often know this in advance as well); example uses such as *Break a Conjecture* are thus rarely employed. Students are also often unable to build upon familiar examples and on their intuition due to their relatively limited mathematical experience. This may be one reason why students seldom demonstrate the meta-awareness that mathematicians demonstrate. Our results point to the power of intentional example exploration in supporting one's understanding of conjectures and their proofs. By better understanding mathematicians' strategies when thinking with examples, we can uncover and elaborate ways to more effectively support students' example exploration and subsequent proof development.

Concluding Remarks

Our findings suggest some implications for the teaching and learning of proof in school mathematics. Mathematicians' practices of engaging in systematic, multi-stage example exploration suggest that students may benefit from learning how to vary their example use and to assess the relative merits of different types of examples when exploring conjectures. Teaching practices that encourage exploration of multiple example types, and that require students to clarify and justify their use of examples, could support a greater understanding of the conjectures. Also, during a classroom discussion, comparing different sets of examples across groups of students could highlight the ways in which some types and uses of examples may be more beneficial than others in supporting both understanding and proof development. In this sense, a stronger understanding of the strategies mathematicians employ as they use examples to develop, explore, and prove conjectures may ultimately inform the design of instructional practices and curricula that effectively foster students' abilities to prove. Mathematicians clearly possess an awareness of the powerful role examples can play in exploring, understanding, and proving conjectures, as well as the ability to implement example-related activity in meaningful ways. Thus, in order for students to develop such awareness and ability, it is important to help them learn to think critically about the types of examples, uses of examples, and associated strategies they can employ as they engage in exploring and proving conjectures. Building on both our findings and on others' prior work, the exploration of students' example use could inform a new approach to conjecture development and proof, one that highlights the power of strategic example-based reasoning and activity.

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