MIDDLE SCHOOL STUDENTS' EXAMPLE USE IN CONJECTURE EXPLORATION AND JUSTIFICATION

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Although students' difficulties in developing and understanding proofs in mathematics is well documented, less is known about how students' example use may support their proof practices, particularly at the middle school level. Research on example use suggests that strategic thinking with examples could play an important role in exploring conjectures and developing appropriate justifications. This paper introduces a framework of middle-school students' example exploration, distinguishing between the types of examples students use and the uses examples play in making sense of and proving conjectures. Drawing from clinical interviews with 20 students, we present thirteen categories of example types and seven categories of uses, followed by a discussion of each set of categories and their connections to one another.

Keywords: Middle School Education, Problem Solving, Reasoning and Proof

Objectives: The Importance of Supporting Proof in School Mathematics

Proof in school mathematics has received increased attention over the past decade, with researchers arguing that it must be a central part of the education of all students at all grade levels (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). Both the *Common Core State Standards for Mathematics* (2010) and the *Principles and Standards for School Mathematics* (2000) argue that a central hallmark of mathematical understanding is the ability to prove, and that the mathematics education of students from pre-kindergarten through grade 12 should enable all students to develop and evaluate mathematical conjectures, arguments, and proofs. Middle school in particular is a critical time for students to develop the ability to reason deductively, resulting in recommendations for curricular and pedagogical changes emphasizing proof in beginning algebra classes (Epp, 1998; Marrades & Gutierrez, 2000).

These recommendations pose serious challenges, however, given that many students struggle to recognize, understand, and produce deductive arguments (e.g., Chazan, 1993; Harel & Sowder, 1998). Researchers have posited that a critical source underlying students' struggles to understand proof is their treatment of examples. On the one hand, students tend to engage in example-based proofs, pointing to a few successful examples as justification that a mathematical statement is true (e.g., Healy & Hoyles, 2000; Porteous, 1990). On the other hand, deliberate exploration of examples is not explicitly supported as a strategy to foster deductive reasoning; students have few opportunities to strategically analyze examples in order to make sense of a mathematical statement or to gain insight into the development of its proof.

We suggest that providing students with opportunities to carefully analyze examples may contribute to their abilities to develop and make sense of conjectures and their proofs. Studies of mathematicians suggest that the process of experimenting with examples is a critical aspect of proof development (Epstein & Levy, 1995). Although scholars have noted a number of potential roles of example use, little research has focused on characterizing these roles with regard to facilitating students' learning to prove. In fact, very little is known about how middle school students think with examples, whether their example use can facilitate deeper mathematical understanding, or whether and how examples can support students' attempts to develop proofs.

This paper presents the results of a study aimed at identifying the roles of middle school students' example use. We introduce a framework that distinguishes between the *types* of examples students use and the *uses* examples play in making sense of and proving conjectures. Our findings indicate that students made use of a variety of example types and used examples in different ways in order to check a conjecture's correctness, convince themselves and others that it held true, better understand a conjecture, and develop justifications to support their statements.

Theoretical Background

One common model of students' mathematical reasoning is that their understanding of mathematical justification is "likely to proceed from inductive toward deductive and toward greater generality" (Simon & Blume, 1996, p. 9). [For this discussion, inductive refers to generalizing from examples, and is not to be confused with mathematical induction, a valid method of proof.] This expected progression is reflected in various mathematical reasoning hierarchies (Balacheff, 1988; van Dormolen, 1977; Waring, 2000) as well as in many curricular programs (e.g., Lappan et al., 2002). However, not only do students find this transition difficult to navigate, studies also suggest that their development may not be as straightforward as the induction-to-deduction model; in fact, students may follow a "zig-zag path" (Polya, 1954) between example exploration, conjecture, proof, and back again (e.g., Ellis, 2007).

One approach to helping students navigate the transition to deductive reasoning involves emphasizing the limitations of examples as proof, thus helping students recognize the need for deductive arguments. It has, however, proven difficult to help teachers leverage this technique in order to successfully foster their students' proof abilities (Bieda, 2011). In addition, this approach positions example-based reasoning strategies as stumbling blocks to overcome. We suggest an alternative stance by positioning strategic thinking with examples as an important object of study in its own right. From this perspective, reasoning with examples is viewed as a potential foundation for the development and understanding of conjectures and proofs.

The Roles of Examples

Examples play a critical role in mathematical practice, and the time spent analyzing particular examples can provide not only a deeper understanding of a conjecture, but also insight into the development of its proof (Epstein & Levy, 1995). The role examples play in the work of middle and high school students, however, is less well understood. Although research has demonstrated students' overwhelming reliance on examples as a means of verification and justification, less is known about how students think strategically with examples.

Research on students' thinking does suggest that examples can have different potential roles and uses. For instance, Buchbinder and Zaslavsky (2009; 2011) introduced four different types of examples (confirming, non-confirming, contradicting, and irrelevant) and examined their status in determining the validity of mathematical statements. Other studies have identified different example types as well, including start-up examples, boundary examples, crucial experiments, reference examples, model examples, counterexamples, and generic examples (Alcock & Inglis, 2008; Balacheff, 1988; Michener, 1978; Watson & Mason, 2001). Studies examining the role of examples in understanding conjectures have found that analyzing structural similarities across examples can support proof development (Pedemonte & Buchbinder, 2011).

This body of research suggests that example use plays an important role in understanding conjectures and potentially supporting the development of valid proofs. However, there remains much to be learned about what types of examples students exploit, particularly at the middle school levels, and how they use them when developing and exploring conjectures. In this study we accordingly characterize the roles and strategic uses of examples in terms of a more comprehensive framework for developing, exploring, and proving conjectures.

Methods

Participants and instrument. Participants were 20 middle-school students (12 sixth-graders, 6 seventh-graders, and 2 eighth-graders), each who participated in a semi-structured 1-hour interview. Eleven students were female and 9 students were male. Seventeen students were in general 6^{th}, 7^{th}, or 8^{th}-grade mathematics courses using the Connected Mathematics curriculum, while 2 students were in algebra and 1 student was in geometry.

The interview instrument presented students with seven conjectures (see Table 1 for sample conjectures). The interviewer asked the participants to examine the conjectures, develop examples to test them, and then, when they could, provide a justification. The conjectures addressed ideas in number theory and geometry that were accessible to a middle-school population, and every conjecture except Conjecture 6 was true. Fifteen out of the 20 participants viewed only the first four conjectures; the remaining 5 participants had extra time to view all seven conjectures, resulting in 95 total responses to code. After the students worked with examples for each of the conjectures, they were asked why they chose the examples they did.

Table 1: Sample Interview Conjectures

Conjecture 1	Eric thinks this property is true for every whole number. First, pick any whole number. Second, add this number to the number before it and the number after it. Your answer will always equal 3 times the number you started with.
Conjecture 4	Bob thinks this property is true for every parallelogram. The angles inside any parallelogram add up to 360 degrees.
Conjecture 6	Kathryn thinks this property is true for every whole number. First, pick any whole number. Second, multiply this number by 2. Your answer will always be divisible by 4.

Data analysis. Coding began by identifying each of the examples students produced for each conjecture. We then developed emergent codes to identify example types and uses. *Types* refer to the different characteristics of examples students used, and *uses* refer to the roles that the examples played in students' investigations. The research group discussed the codes and clarified uncertainties as emergent codes solidified. Codes to determine example types depended on the participant's discussion of the example, rather than on a determination based only on the example itself. For instance, the same number, 1, could be considered "common" from one student's point of view or a "boundary case" from another student's point of view. Furthermore, the same example could be coded in multiple ways based on the participant's explanation. Three different researchers on the project team coded portions of the conjecture responses so that each conjecture response was ultimately coded independently by at least two different team members.

Results and Discussion

We found 13 categories of example types (Table 2) and seven categories of example uses (Table 3). Each table introduces the category name with the number of instances in which the example type or use occurred in the data set, its definition, and a representative example to illustrate each example type and use. We discuss the example types first, and then present example uses.

Table 2: Types of Examples

Example Type	Definition	Data Example
Dissimilar Set (41)	An inclusion of examples that are all different from one another.	"I tried to pick both prime numbers…composite numbers and then odd numbers and even numbers. So just have a variety of different kinds of numbers."
Generic (28)	An example with unimportant specific characteristics. These examples demonstrate a general idea about the conjecture.	[Uses 50 to explain why Conjecture 1 works]: "So this one [51] is 1 more, and this one [49] is 1 less. So if you take this one [the 1 from 51] over to that one [49] it turns into 50. And then that one goes into 50, and 50."
Common (24)	An example described as typical or one that many would think of.	[After choosing 10 and 15 as example cases]: "I picked a more uncommon number [15] and a more common number [10], and they both worked out."
First Thought Of (22)	Example is the first that came to mind; no evidence of thoughtful decision about example selection.	"I just…kinda what popped into my head."
Unusual (21)	An example that does not occur often and may have odd or strange characteristics.	[To explain using 1,028 as an example]: "Well, the 1028's kind of like a little stand-out number because it's – it's large."
Random (12)	Example is arbitrarily chosen, with the "randomness" intentional to highlight the likelihood of the conjecture being true.	This code refers to what students consider random, rather than true mathematical randomness: "For a problem like this you want to pick random numbers. Not selected numbers."
Conjecture Breaking (11)	Example chosen to disprove the conjecture; counterexample.	"I wanted to do numbers that were hard for it…it was less likely for them to be divisible by 3, I think."
Easy (11)	Examples that are easy to operate on or compute with.	[*Prompt: So why did you try it out with 15?*] "Because it's an easy number to use."
Known Case (9)	Student picks an example in which properties or features pertaining to the conjecture are already known.	"A rectangle is a parallelogram, so that is four 90 degree angles, which is 360."
Boundary Case (5)	An extreme example or a special case example, such as the identity.	[Explaining the use of 0]: "You kind of have to try it with every not possibility like, not like 3, you know, to, you know, 100, but kind of like get down to the origin of the number. Like 1 and then, you know, 0."
Similar Set (3)	Deliberate inclusion of examples similar to one another.	[*Prompt: Are these two (triangles) different or similar from each other?*] "Similar, because they have same numbers in two of the sides and different numbers in this side."
Progression (2)	First one type of example is chosen, then student deliberately switches to a different type, and may continue switching to new types.	"You would first test a typical number to just see, like, okay in general was this going to be true. And then if that – if he was true on that, then you say, okay, then I would test a more unusual number to just, like, to test his property."
Favorite (2)	Example represents a favorite number or shape.	"The only reason I picked 6 is because that's my lucky number."

Unsurprisingly, we found evidence that students chose example types that were not always deliberate or thoughtful: For instance, the categories *first thought of*, *favorite number*, and *easy* represented example types that were not necessarily connected to the content of the conjecture at hand. These example types also did not typically support the development of deductive arguments. However, these categories only represented 18% of all of the example types. When examining the participants' discussion of their examples, we also found many cases of deliberate and thoughtful example choices, which we discuss below.

By far the most prevalent type of examples was the *dissimilar set* type; many participants indicated a belief that choosing a variety of examples was a more reliable method of testing a conjecture. For instance, one student explained the importance of choosing a dissimilar set:

> You should find numbers that maybe aren't as alike to test just so you have all kinds of differences covered. Like when you're maybe testing students for a survey, you want to have as many different students and maybe different race, different families, and everything. Just a bigger background so maybe you'll get more accurate information.

The two students who used a *progression* of example types demonstrated similar reasoning by first picking common numbers, then deliberately shifting to less common numbers.

The inclusion of a dissimilar set often resulted in a discussion about the importance of picking both *common* and *unusual* examples. Some students indicated that unusual examples are more convincing than other types of examples. For instance, one student, Eva, tested a conjecture that every even number added to half of itself would be divisible by 3. She explained that she deliberately tested numbers that could only be divided by 2, such as 10. Eva tested those unusual numbers because "it was less likely for them to be divisible by 3, I think." It is worth noting that in this case, a number such as 10 was unusual in Eva's eyes in relationship to the conjecture, even though 10 might not be an unusual number for her in general. Unusual examples and boundary case examples both played an important role: they could lead to *conjecture breaking* example types, and they were particularly convincing because if a conjecture held for an unusual or extreme example, it may be more likely to be true overall.

There were some example types that were more strongly connected to proof development, such as *known cases* and *generic* examples. Although deductive arguments were not solely developed through these example types, known case examples and generic examples helped students reason through the structure of the conjectures. For instance, Rodrigo examined Conjecture 4 (Table 1), and in order to better understand the conjecture, he began with a known case example, the rectangle. Rodrigo knew that the conjecture held true for a rectangle: "A rectangle is a parallelogram, so that is four 90-degree angles, which is 360." He then took the rectangle and adjusted it to think about how a new example would work with the conjecture (Figure 2):

Figure 2: Rodrigo's adjustment of the "known case" rectangle into a new parallelogram

After examining the new example, he said, "Oh yes it would work for every one – this doesn't really matter any more." Developing a generic example, Rodrigo explained further: "That's just a random drawing. [Figure 3]. It is a rectangle in disguise because you cut this off and you put this, over here, ta da! And it becomes a rectangle. And rectangles, well, see, equals 360."

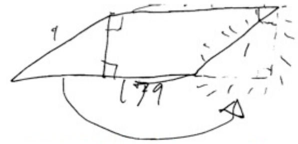

Figure 3: Rodrigo's generic example

By "random", Rodrigo indicated that the particular nature of his example was not important because it illustrated a more general point; hence, this example type was coded as *generic* rather than *random*. Across the participant group, it is notable that 15% of the example types were generic in nature; it was the most prevalent code, second only to *dissimilar set*. In general, the types of examples students chose in order to foster their understanding of the conjectures suggest that middle school students can and do engage in deliberate and strategic example choices.

Example Uses

Students also demonstrated a variety of example uses. Each of the seven categories in Table 3 includes a frequency, a definition, and a representative data example.

Table 3: Example Uses

Example Use	Definition	Data Example
Check (69)	Student selects examples to test whether the conjecture holds.	"Just, you know, test, just to see if it actually does work or not."
Support a General Argument (28)	Student uses a generic example to describe a more general phenomenon in support of a deductive proof.	"When you're taking half of it, then that number is, because it ends up being thirds. So it's always going to be true because if you do…514. That's always going to be 1208, which means that it's broken up into thirds, so no matter what it's going to be divisible by 3."
Convince (25)	After checking the conjecture, student tries additional examples in order to convince oneself or others that the conjecture must be true.	"You can't be sure if you only test one number because one number, because in almost every case there is exceptions to the stuff if it's not true."
Understand (21)	Student uses an example to make sense of the conjecture; may lead to insights that support deductive proof.	"Let's try 5…okay. Those are the two that I needed. Now I kind of know the logic behind it."
Asked (19)	Student was asked to choose and example; the only evidence that a student produces an example is because s/he was explicitly asked to do so.	S: "I'm totally convinced it's true." I: "You don't even need to – do you need to test out any examples?" (Student shakes head.) I: "Okay. Let's say that you didn't know it was true. Are there any kinds of rectangles you would want to test it out on?"
Support Empirical Proof (9)	Student offers examples as a justification of the truth of a conjecture.	I: "Say that you wanted to show that this was always true." S: "I would use these examples, and probably a few more."
Disprove (6)	Student tests an example in an attempt to disprove the conjecture.	S: "Any whole number? Oh, I thought you just meant even numbers. I wouldn't think that's true then." (Tries 9 to disprove). "Nine times 2 equals 18. 18 divided by 4 equals question mark."

The most prevalent use of examples was in *checking the correctness* of a conjecture; 39% of the example use instances occurred when students used examples to test conjectures. Part of this prevalence may be due to the fact that students were encouraged to test examples during the interview. Checking correctness occurred with many different example types, ranging from the first number thought of to unusual examples to dissimilar sets. Among the other example uses, there were some connections between how students used examples and the types of examples they employed. The strongest connection was between generic examples and the *support a general argument* use. This link is unsurprising because the purpose of a generic example is to illustrate a broader point. Similarly, using examples to *disprove* a conjecture typically relied on conjecture breaking example types, but also occasionally made use of boundary cases or unusual examples.

Another set of links emerged when students used examples to convince and understand. The example types that students viewed as more convincing, such as dissimilar sets, unusual examples, and boundary cases, were often the ones they employed when continuing to further check examples after an initial test. For instance, Alyssa tested Conjecture 1 with the number 4, and found that it worked. She then explained that she was not convinced: "I think I need to try it a few more times to make sure." She indicated that she should try different numbers, such as both even and odd numbers, in order to really be sure the conjecture would work. While testing a dissimilar set of examples in order to further convince herself that the conjecture held true, Alyssa began to use the examples to understand the structure of the conjecture. Through this process, she was then able to produce a general argument, using the initial example, 4, as a generic example: "When you add the number before and the number after, those two numbers will equal twice the first number I guess. Because, like, for 4 + 3 + 5, if you drop one off the 5…then 3 would kind of be 4. So it'd be 4 + 4 + 4. Which would be, like, 12, or 4 times 3." Alyssa's general argument was not unusual amongst the 20 participants; we coded students' justifications as part of a larger study and found that after exploring examples, students who attempted justifications were able to produce deductive arguments or valid counterexamples a little over half the time.

It is worth noting that in 19 responses, students did not see a need to produce an example at all; this typically occurred because the student already believed the conjecture to be true, and therefore not in need of testing. For example, Andre was asked to consider the conjecture that for any triangle, the sum of the length of any two sides are greater than the length of the third side. Andre did not see a need to test this property because "That's a property already proven by the, you know, the community." This finding is in contrast to previous results suggesting that students want to test conjectures even when presented with their proofs (e.g., Chazan, 1993).

Conclusion

This study presented a framework of the example types and example uses middle school students employed when making sense of, exploring, and attempting to prove conjectures. Our findings support earlier studies suggesting that students' uses of examples can play an important role in exploring and understanding conjectures, as well as in potentially supporting the development of valid proofs (Alcock & Inglis, 2008; Buchbinder and Zaslavsky, 2011; Pedemonte & Buchbinder, 2011). Moreover, our study suggests that distinguishing between example types and example uses may be an important component in better understanding students' thinking with examples; this distinction can also provide a potential structure for more in-depth analysis of how example type may be linked to example use in future studies.

One compelling finding was that many of the students who explored conjectures with multiple examples were able to produce deductive arguments, valid counterexamples, or general arguments that relied on generic examples. These results run counter to the many studies demonstrating K-16 students' difficulties in producing valid mathematical proofs (e.g., Chazan, 1993; Healy & Hoyles, 2000). The fact that the students were able to produce valid arguments after in-depth example exploration provides initial evidence that strategic and thoughtful use of examples can indeed support the development of mathematically appropriate proofs, even at the middle school level. This suggests the importance of continuing to study the roles examples can play in supporting middle-school students' learning to prove.

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