Understanding the Connections between Students' Natural Way of Reasoning and Mathematical Ways of Reasoning

There are two kinds of reasoning, as we said: demonstrative reasoning and plausible reasoning. Let me observe that they do not contradict each other; on the contrary they complete each other.
-G. Polya, *Induction and analogy in mathematics*, 1954 p. vi

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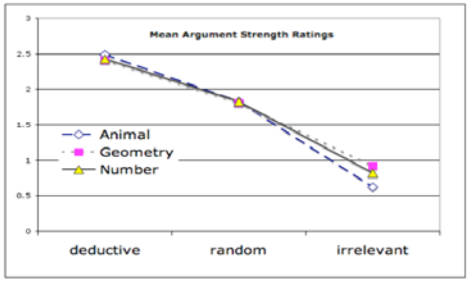
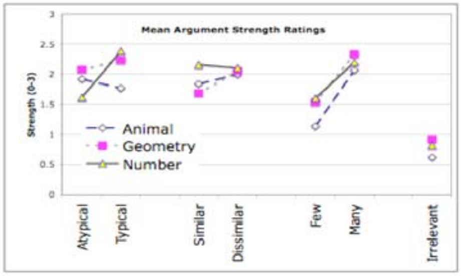
Motivation for Study: All humans make inductive inferences constantly and without formal instruction. Developmental research demonstrates that even very young children have some remarkably sophisticated inferential strategies. Yet, in contrast to the ubiquitous nature of, and children's facility with, inductive reasoning, deductive reasoning typically occurs only in restricted contexts (e.g., mathematics classrooms) and is often a difficult skill for children to acquire. In the world outside of the mathematics classroom, it is common and reasonable to develop facts and ideas via empirical generalizations and causal theories. Thus it makes sense that students bring to bear their natural ways of reasoning as they encounter mathematical ideas and problems. Students' natural ways of reasoning, however, are not sufficient for establishing facts and ideas with certainty in the domain of mathematics. **Hence, the overarching question motivating this project is whether we can elaborate strategies used for inductive reasoning in both non-mathematical and mathematical contexts, and whether those inductive strategies can be built upon to help students develop deductive, mathematical arguments.**

Abstract: Despite the growing emphasis on proof in school mathematics, research continues to paint a bleak picture of students' abilities to reason mathematically. In contrast, recent cognitive science research has revealed surprising strengths in children's ability to reason in non-mathematical contexts. Accordingly, **we examined the connections between students' reasoning in mathematical and natural contexts**. Our study was designed to measure how convincing the students found different forms of evidence in three areas: numerical relations; geometry; and natural science. We found two themes consistent across areas: that **5 examples were always more convincing than 1**, and that **irrelevant examples were always less convincing than other examples**. The evidence also indicates that while typical examples for numerical relations and geometry were generally more convincing, in natural science the atypical examples were slightly less convincing. Furthermore, in geometry and natural science, dissimilar examples were more convincing than similar, while in numerical relations the results were less conclusive.

Methods:
- 47 Middle school mathematics students
  - 1 6th-grade student, 26 7th-grade students, and 20 8th-grade students
- Paper and pencil surveys
  - Statements paired with 5 different types of evidence
    - Individual rating on a 4 point Likert scale (ranging from "not at all convincing" to "fully convincing")
    - Ranking in relation to each other
- **Each survey contained irrelevant, random, and deductive evidence items, as well as a single set of paired theme items: similar and dissimilar evidence; typical and atypical evidence; and 1 example and 5 examples.**
- Analysis across the three areas and 9 qualities of evidence





Conclusions:
- Tasks were within the capacity of middle school-aged students
- More examples were consistently better than fewer
- Irrelevant examples were consistently weaker than relevant
- Results for similarity and typicality were mixed
- Some evidence of domain differences
- . Evidence quality with most confidence (one examples versus five examples)
  - fewer than 2/3 of the students showed a distinction for mathematical items
  - 80% showed the distinction for animal items.

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	SIMILAR vs. DISSIMILAR	IRRELEVANT	DEDUCTIVE

Numerical Relations

Bennett says, "If you add any three consecutive numbers, the sum is equal to three times the middle number."

Similar: Beth added three sets of consecutive numbers that she thought all looked pretty similar, and it worked each time.

23 + 24 + 25 = 72 3 ∙24 = 72
12 + 13 + 14 = 39 3∙13 = 39
32 + 33 + 34 = 99 3∙33 = 99

Dissimilar: Franz added three sets of consecutive integers that were different, and it worked every time.

7 + 8 + 9 = 24 3∙8 = 24
113 + 114 + 115 = 342 3∙114 = 342
54 + 55 + 56 = 165 3∙55 = 165

Irrelevant: Tom added some numbers that weren't consecutive.

2 + 5 + 6 = 13 3∙5 = 15
10 + 11 + 14 = 35 3∙11 = 33
23 + 32 + 33 = 88 3∙32 = 96

Deductive: Jo explains that Bennett is right, and says that she can show you with blocks. She starts with 9 blocks, and puts them in three stacks:

2 + 3 + 4 = 9

Then Jo takes one block from the 4 stack, and puts it on the 2 stack. Now all the stacks have three blocks.

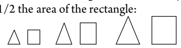
3 + 3 + 3 = 9

Jo says that you can always take one block from the biggest of the stacks, and put it on the littlest, to make this work.

Geometry

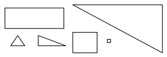
Timon says, "The area of a triangle is <u>always</u> exactly half the area of a rectangle with the same base and height."

Similar: Rob looked at the following three pairs of triangles and rectangles because they were pretty similar, and in each case, the triangle's area was always 1/2 the area of the rectangle:



Dissimilar: Courtney looked at the following three pairs of triangles and rectangles because each pair was different from the other pairs, and in each case the triangle's area was always 1/2 the area of the rectangle:

Irrelevant: Mariella looked at three rectangles and three triangles that did not have any of the same bases or heights. They all had different areas.



Deductive: Valerie argued that a triangle has to have half the area of a rectangle with the same base and height, because you can take any triangle of any shape, and make a rectangle that has the same base and height like this:

Since the area of a rectangle is b x h, and the area of a triangle is1/2 b x h, then any triangle will have an area that is 1/2 the rectangle.

Natural Sciences

Therese says, "<u>All</u> insects have chitinous skeletons."

Similar: Alexis looked at ants, roaches, and beetles because, she said, they're all pretty similar. She found that all three have chitinous skeletons.

Dissimilar: Virginia looked at ants, wasps, and butterflies, because, she said, they're all different from each other. She found that they all have chitinous skeletons.

Irrelevant: Jose looked and found that snails, worms, and dogs do not have chitinous skeletons.

Deductive: Chuck explained that chitin is the substance that forms a hard exoskeleton in all "arthropods" (or invertebrates with hard exteriors). Since all insects are defined as arthropods, all insects have chitinous skeletons.

Future Directions:
- Diagnose students' representations of categories, similarities, and typicalities in mathematical domains
- Examine middle-school, non-STEM college, and STEM graduate students
- Empirically measure similarities and typicalities in numerical relations and geometry (this work was based on our intuition and best guesses)

Example Triad Similarity Measurement Item:

Which of the following is more similar to 101:
99 105